

Characterizing rifle performance using circular error probable measured via a flatbed scanner

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ABSTRACT. Serious shooters, whether shooting firearms or air rifles, often want to determine the effects of various changes to their equipment. Circular error probable offers an important, quantitative measure that complements center-to-center techniques that are frequently used. Flatbed scanners coupled with simple analytic approaches provide accurate, reproducible, quantitative methods for determining optimal equipment configuration and assessing the effects of changes in technique.

1. Introduction

Shooters, whether of firearms or air rifles, often want to make comparisons of accuracy among various configurations of their equipment. Examples include different loads for cartridges, different types of pellets for air rifles, and different barrels. Reproducible, quantitative comparisons allow unambiguous conclusions regarding which configuration provides the best accuracy. In addition to determining the ultimate accuracy of equipment, the shooter may also wish to quantitatively measure improvements in shooting technique.

In this paper, we outline a simple method for using circular error probable¹ (CEP) measured via a flatbed scanner and a specially designed target to make measurements of the point-of-impact (POI) of a projectile with accuracies that are typically better than 0.1 mm.

We recognize that aspects of this discussion will, of necessity, be technical. Hence, we have included an appendix containing definitions and notes that readers who are less familiar with some of the mathematics may find useful.

Center-to-center² (c-t-c) measures for small groups of shots are frequently used by shooters to compare the accuracy of various in configurations. While these types of measurements certainly have their place, they are often done with very small numbers of shots. In addition, the human tendency to report “the best results I’ve ever gotten” as “typical results” may lead to optimistic assessments of some configurations. At their worst, methods using small numbers of shots and ignoring most of the information from those shots have a substantial probability of leading the shooter to choose the less accurate configuration. Using the method outlined in this paper, we will show how to estimate circular error probable and how this measure compares with various c-t-c measures using small numbers of shots.

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¹As will be discussed in the following sections, circular error probable is the radius of a circle that is expected to contain half of the shots fired.

²It is assumed that most shooters are familiar with center-to-center measures. However, for completeness, center-to-center is defined as the maximum distance between the centers (points of impact) of a set of shots.

Finally we will use the method in a few worked examples to illustrate the types of decisions that a shooter can make in optimizing equipment.

2. Why consider CEP?

Given that many shooters are comfortable with characterizing their shooting systems using center-to-center (c-t-c) techniques, the obvious question is, “why add another technique to the repertoire?” In order to answer this question, we begin by comparing c-t-c with CEP to see similarities and differences. We will then lay out reasons why CEP may in fact be preferable as a measure for rifle system performance.

CEP is defined as the radius of a circle containing the point-of-impact (POI) for 50% of shots. Below, we will outline various methods for estimating CEP. However, in anticipation of this discussion, we will compare c-t-c and CEP as methods for determining which of two system configurations is better using a Monte Carlo simulation. We are in full agreement with Press et al. when they say, “Offered the choice between mastery of a five-foot shelf of analytical statistics books and middling ability at performing statistical Monte Carlo simulations, we would surely choose to have the latter skill.”[1] Thus, we will make use of Monte Carlo simulations as a way of estimating the effects of different measures.

As a first study, consider the comparison of two system where the POI is normally distributed about the aim point in x and y and the standard deviation is the same in both the x and y directions. We examine a range of differences in the underlying standard deviations of the two systems. We use both c-t-c and CEP to distinguish whether the system that is known to be better is assessed to be better via the metric in question. In practical terms, this might be two rifles that are being compared, two loads for the same rifle or two types of pellets for an air rifle. While the applications vary, the type of comparison is general.

The study was done using approximately 1500 randomly generated samples drawn from normal distributions having ratios between the standard deviations of the two systems as shown by the column labeled Sigma Ratio in Table 1. These samples were then taken N at a time and compared in pairs. In the case of the c-t-c and the CEP columns, if the sample drawn from the distribution with smaller standard deviation had a larger measure than that drawn from the distribution with the larger standard deviation it was considered an error. These columns show the error rate. Thus, when the standard deviations differ by a ratio of only 1.2, the error rates for either c-t-c or CEP are nearly 1 in 3 – not very good. As the number of samples increases, the error rates go down for both measures but neither measure has a clear advantage. While we only calculated CEP measures for $N = 20$, it is probably the case that the c-t-c measure is not too different.

If comparison of which measure is larger were the only possible operation in trying to decide which system is better, there would be little argument for CEP. However, because estimates of CEP are based on means and standard deviations, there are simple, well understood statistical tests that can be brought to bear to improve our inferences. In this case, we have used a test based on the mean, the t-test, to improve our inferences. Were the data on which this table has been based to have been acquired experimentally, we would not know the underlying means and standard deviations. Hence, the only estimates for the mean and standard deviation would be those that could be derived from the experimental data. If we require that the means of the radii about the aim point differ at the 90% confidence level as determined by the t-test and that the CEP from the distribution with smaller sigma be larger than the CEP from the distribution with larger sigma to generate an error, we see that this statistical test reduces our error rate by an order of magnitude. This advantage comes at a price. When the difference of the means is less than that required to reach the 90% confidence level, we can only “fail to reject the null hypothesis” that the distributions are the same. In order to really discriminate between distributions where the sigma ratio is small, we must get more samples.

A simple but valid conclusion to carry away from the first portion of this discussion is that if we are looking for small differences between system configurations, it will be necessary to collect a substantial

Error rates when using c-t-c and CEP as metrics											
Sigma Ratio	N = 3			N = 5			N = 10			N = 20	
	c-t-c	CEP	t-test CEP	c-t-c	CEP	t-test CEP	c-t-c	CEP	t-test CEP	CEP	t-test CEP
1.2	0.36	0.35	0.04	0.34	0.33	0.03	0.28	0.22	0.02	0.13	0.01
1.4	0.27	0.27	0.02	0.21	0.20	0.02	0.19	0.12	0.01	0.01	0.00
2.0	0.11	0.10	0.00	0.04	0.03	0.00	0.01	0.00	0.00	0.00	0.00
2.8	0.05	0.04	0.00	0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.00
3.5	0.02	0.02	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

TABLE 1. Comparison of error rates using c-t-c and CEP as measures. $\sigma_x = \sigma_y$, the Sigma Ratio indicates the ratio of standard deviations between the two systems under consideration. Major headings indicate the number of samples used for the calculation. Columns labeled t-test demonstrate the order of magnitude improvement that is possible by adding standard statistical tests to the decision making process.

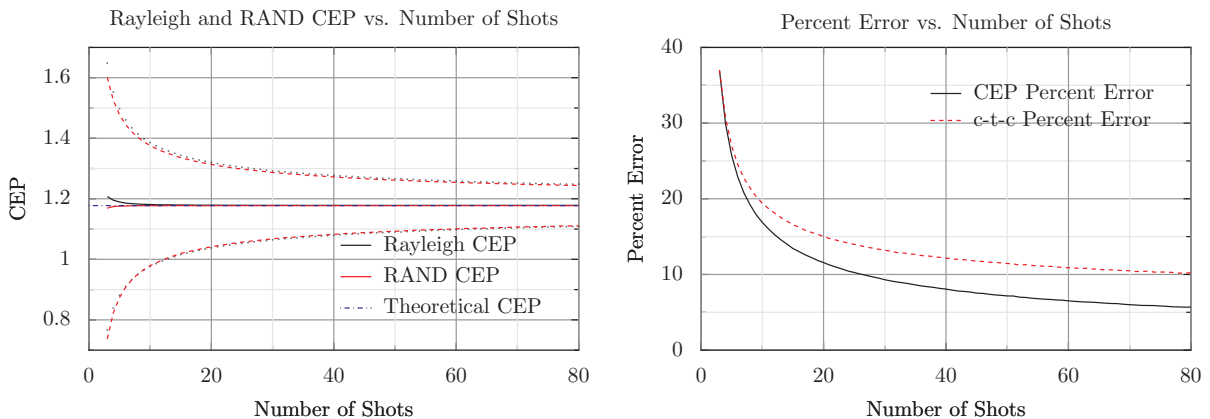


FIGURE 1. Both Rayleigh and RAND estimates of CEP as a function of the number of shots used for the measurement. 1σ error bars are shown for both. These estimates are based on a million sample Monte Carlo simulation. Both measures have been corrected for bias at small numbers of shots. These corrections are small given that the uncertainties are dominated by the number of shots for small N . Note that both measures rapidly approach the theoretical value and are relatively independent of the number of shots. In addition, note that the percent error for c-t-c shown in the right-hand graph is always larger than the percent error for CEP. By the time the sample size is 20, the differences are becoming non-negligible.

number of data points – three and five shot groups will only discriminate between fairly large differences.

In our view, there are five key reasons for shooters to add CEP to their toolkit for evaluating systems.

- CEP is relatively constant as a function of the number of shots fired, as shown in Table 2 and Figure 1. Thus, it serves as a better metric for characterizing a system. On the other hand, c-t-c increase approximately linear when plotted versus the logarithm of the number of shots as shown in Figure 2.
- As shown in the right half of Figure 1, the percentage error in the CEP measures is always less than that of the c-t-c measure regardless of the number of shots used in the analysis. At the 20

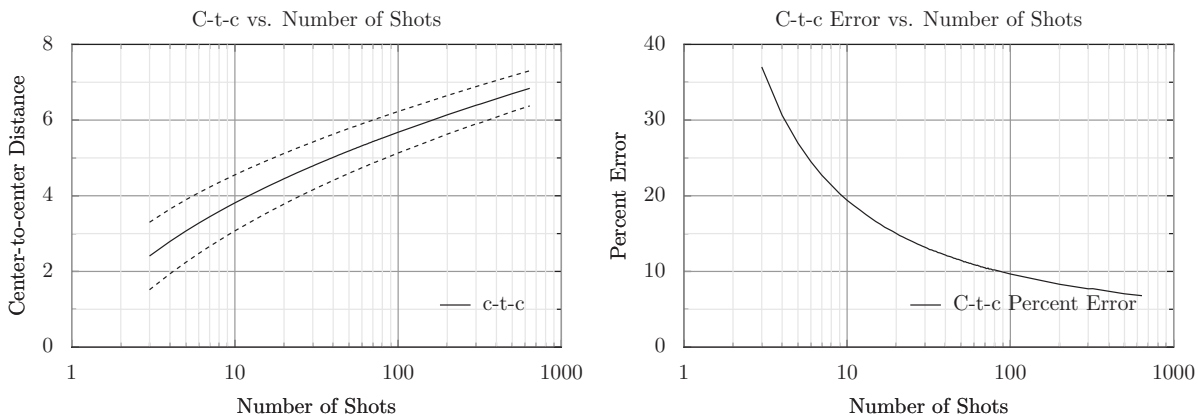


FIGURE 2. C-t-c as a function of shot number based on a million sample Monte Carlo simulation. Note that shot number is plotted on a log scale. 1σ error bars are shown. Note that c-t-c is approximately linear in $\log(N)$. Thus, even as the number of shots in the group approaches 1000, the size of the groups are still growing. This is because larger groups sample further into the tails of the Gaussian distribution and the max operation contained in the c-t-c is dominated by these samples in the tails.

shot level we use in this paper, the differences is becoming noticeable.

- CEP can be easily used to predict expected (or detect unexpected) performance on circular targets. In addition, it is easy to extend the concept from the 50% level to the 90% and 95% levels. Thus the shooter can anticipate the size of circle into which a specified fraction of shots should fall and know whether to take corrective action or not.
- Because estimates of CEP are based on commonly used statistical measures such as the mean and standard deviation, the statistical tests for drawing inferences are well developed and widely available. We will illustrate this by using Student's t-test and the F-test. While these are not the only such tests available, they illustrate the point.
- CEP and the underlying data used to compute it provide ready access to sources of variability in the system. Thus, experiments can be devised to isolate these sources of variability and potentially control them.

3. Estimating CEP

CEP has been used routinely for estimating the effectiveness of bombs and ballistic missiles. Since the early 1950's tables for estimating CEP have been readily available [2]. While there is a substantial literature on estimating CEP [2] – [8], there is considerable confusion on the Web about how to make the estimates.³ The interested reader should see the original literature for details on the estimation techniques. Because we anticipate that most shooters will be doing their analysis with of order 20 or fewer shots, the uncertainties imposed by the number of shots will be on the order of 10%. Hence, worrying excessively about differences of a few percent is not worthwhile. Rather we present two techniques that

³For example, the Wikipedia [15] entry as of November 10, 2008 indicates that

$$\text{CEP} \approx \sqrt{\text{var}(x) + \text{var}(y) + \text{covar}(x, y)}$$

However, this cannot possibly be correct. If $\sigma_x = \sigma_y = 1$ and they are uncorrelated, this formula gives $\text{CEP} = 1.414$. However, under these conditions, CEP is known [3] to be 1.1774, a result that can be quickly confirmed via a Monte Carlo simulation.

Average c-t-c and CEP's for $\sigma = 1$ at different sample sizes					
N	c-t-c	St. Dev.	CEP Rayleigh	CEP Rand	St. Dev.
3	2.41	0.89	1.20	1.17	0.43
5	3.07	0.86	1.19	1.17	0.30
10	3.81	0.74	1.18	1.18	0.20
20	4.45	0.67	1.18	1.18	0.14

TABLE 2. Comparison based on a Monte Carlo simulation of c-t-c and two different estimates of CEP as a function of the number of shots used for the comparison. Note that for small N , the standard deviations used to calculate the CEP Rand are biased low. In this study, these have been corrected by dividing the measured standard deviations by $1 - \frac{1}{4N} - \frac{7}{32N^2}$. Similarly, the Rayleigh CEPs have been biased corrected by the square of this factor. The standard deviations for both c-t-c and the two estimates of CEP are also shown. The standard deviations for the CEPs do not differ appreciably, hence only one is shown. As would be expected, the standard deviation of the CEP is proportional to $1/\sqrt{N}$.

earlier studies have to found to be adequate that share the virtue of being computationally simple as well as relying on means and standard deviations in ways that make application of the t-test and the F-test obvious.

In this analysis, we will calculate CEP about the mean point of impact. It is assumed that this point can be adjusted via scope settings so that the bias can be ignored.

3.1. Rayleigh. This simple estimator has been suggested by [5] and is based on the mean distance of the points of impact from (\bar{x}, \bar{y}) where (\bar{x}, \bar{y}) represents the average over all the points of impact in both the x and y directions. If we represent this average radius by \bar{r} then

$$(3.1) \quad \text{CEP} \approx 0.9394\bar{r}$$

By measuring the x and y locations for each point of impact, it is trivial to find (\bar{x}, \bar{y}) and then \bar{r} .

3.2. RAND. Fits to the Rand tables [2] have been done. These fits provide a second simple algebraic approximation for CEP based on the standard deviations in the data. Let S_x and S_y be the sample standard deviations for the x and y values respectively and let ρ be the correlation between x and y . If $\rho > 0$ it is always possible to define a new coordinate system in which x and y are not correlated. In this new coordinate system[7], the uncorrelated sample standard deviations are

$$(3.2) \quad S_s = \sqrt{\frac{S_x^2 + S_y^2 - \sqrt{(S_x^2 - S_y^2)^2 + 4\rho^2 S_x^2 S_y^2}}{2}}$$

$$(3.3) \quad S_l = \sqrt{\frac{S_x^2 + S_y^2 + \sqrt{(S_x^2 - S_y^2)^2 + 4\rho^2 S_x^2 S_y^2}}{2}}$$

Given these definitions⁴ for S_s and S_l , the smaller and larger standard deviations respectively, a second estimate of CEP is given by

$$(3.4) \quad \text{CEP} \approx 0.614S_s + 0.563S_l$$

⁴It should be noted that the formulas for S_s and S_l given in [3] differ from those above. Those given by [3] cannot be correct simply on dimensional grounds. However, it is of note that [3] shows the program he used for his analysis. The formulas used in the program match those given here and are dimensionally correct.

Ellipticity	P(50)	P(90)	P(95)
1.0	1	1.88	2.13
1.5	1	1.93	2.25
2.0	1	2.01	2.41
2.5	1	2.11	2.44
3.0	1	2.17	2.48
3.5	1	2.28	2.76
4.0	1	2.39	2.80

TABLE 3. Multipliers to be applied to CEP, denoted in this table by P(50), to determine the radii in which 90 and 95% of the shots will fall as a function of ellipticity. These estimates of the multipliers were determined using Monte Carlo simulations.

If we define the ellipticity of the pattern as $c = S_l/S_s$, this approximation has been shown[3] to be good for $1 \leq c \leq 4$.

It may be desirable to estimate the radius of a circle containing 90 or 95% of the shots. As shown by Harter[4], these radii will be a function of the ellipticity of the pattern. Using the definition of ellipticity above, we show, in Table 3, the multiplier to apply to the CEP in order to find these radii which are denoted P(90) and P(95) respectively.

Since these multipliers are roughly linear in ellipticity, we can approximate the results of the table as follows:

$$(3.5) \quad P(90) \approx (0.17c + 1.68)CEP$$

$$(3.6) \quad P(95) \approx (0.22c + 1.91)CEP$$

These two estimates, one based on the mean radius, the other on the sample standard deviations provide an adequate basis for demonstrating the method and drawing inferences about system performance.

4. Description of method

A flatbed scanner is a common addition to many computer systems. If not already a part of the system, scanners that are surprisingly accurate are readily available for less than \$100. Thus, these are attractive as measuring devices to help a shooter characterize both equipment and shooting technique. The use of a flatbed scanner to assess POI is not a new concept, software such as ‘‘On Target’’[9] is readily available to perform functions such as group analysis. We believe that it is possible to build on this concept and to provide additional information to help the shooter improve his system.

4.1. Resolution. Flatbed scanners typically provide the ability to scan documents at resolutions as high as 600 and 1200 dots-per-inch (dpi). This means that individual pixels are 0.042 mm and 0.021 mm respectively. As will be shown below, these dimensions are not perfectly accurate, however they are surprisingly good given the low cost of scanners.

Most ink-jet printers can be operated in similar resolution ranges. Thus, many shooters already have very precise systems for producing targets and measuring them.

4.2. Target design. Accurate measurements on paper targets demand the ability to register images collected from multiple scans. Because of the construction of both printers and scanners there is no guarantee that the image on the paper or the positioning of the paper in the scanner will be registered at the pixel level. Thus, we have designed a target with registration points that are easy to measure. These provide both the means for registering images as well as for setting the scale in the images. In addition

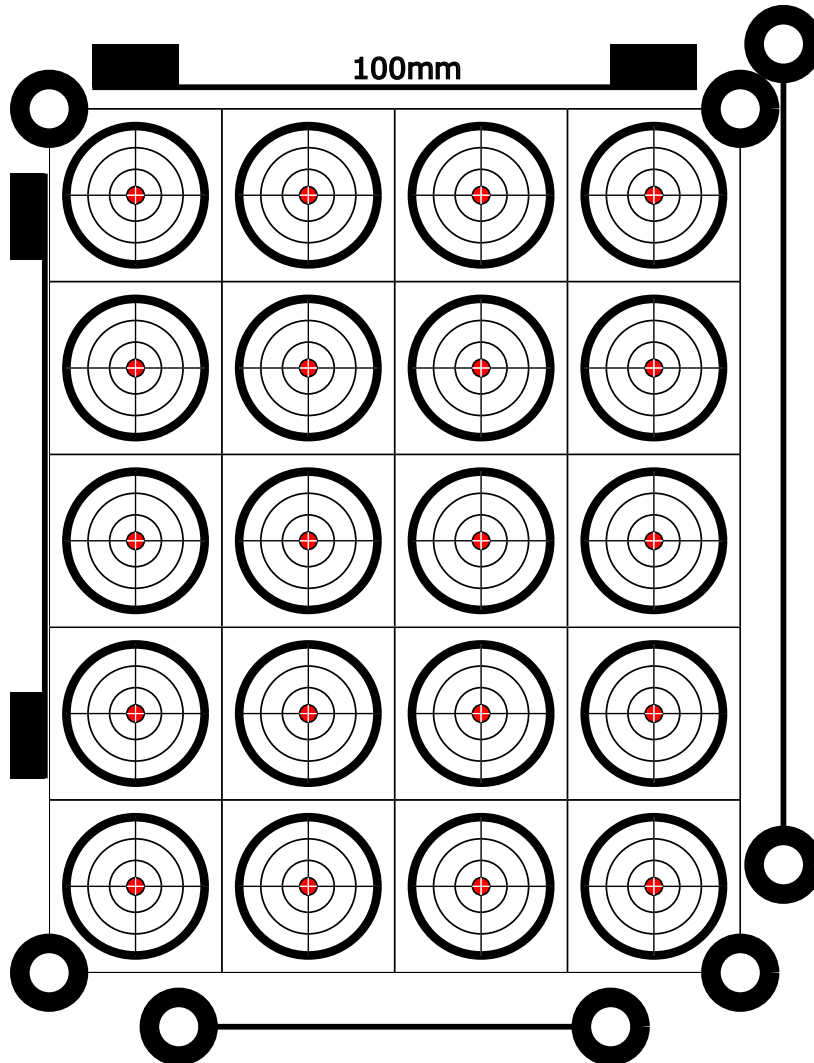


FIGURE 3. Target designed for metrology on a flatbed scanner.

to preventing shot overlap from interfering with the measurement, individual targets for each shot enable correlation of accuracy with known information about each projectile and cartridge.

The targets were designed using Scalable Vector Graphics (SVG), a language for generating vector graphics within XML. The SVG source for our target is available from [10] and is shown in Figure 3.

The four circles at the corners of the target area are designed for registration. When the scan is

measured, their centers can be determined to sub-pixel accuracy. Thus, they can be used for registration.

The bars at the top and left of the image are each 100mm long. These can be measured to ensure that the software printing the image is not scaling the image by substantial amounts in either direction. The pair of circles on the bottom and far right provide additional measures of scaling should these be needed. In practical use, these last sets of circle have not been required, nevertheless, they have been left in the system should later checks be needed.

4.3. Measurement technique. As a way of quickly prototyping this system, we have chosen to use software that is readily available. For mensuration of the image, we have chosen ImageJ [11], software that is freely available from the National Institutes of Health and has a set of measurement tools that meet the needs of this project.

The target can be scanned using any convenient software [12]. We have found that scanning the image in color can sometimes help in resolving uncertainty in the POI. For our experiments we have used scans at 600 dpi and have stored the images as high-quality JPEG images. The compression introduced by saving the images in JPEG has not had a serious effect on the quality of our measurements.

The image is registered to the reference image by first converting the image to binary form – assigning each pixel to either 0 or 100% – and measuring the eight reference circles in the image. The measurements are made using the circular measurement tool and looking at the “center of mass”, the mean of the locations weighted by the image intensity. The effect of this type of measurement is to make the measurement insensitive to the location of the circle tool as long as the entire circle of the tool is in the black area; it sums over only the white area. This locates the center of the circle to a fraction of a pixel. It has been found that in the black region there are occasional specks that are not completely black, thus, the process of binarization helps to ensure that the location of the centers of the circles is not sensitive to the placement of the measurement tool. A detailed step-by-step summary of the measurement procedure is given in Appendix 2 for those wanting to use this technique.

In order to compare the points-of-impact with the aim point, it is necessary to measure a reference image and record the locations of each of the aim points. Once the images are registered, the difference between the aim points and the POI can be measured. The reference image is measured in the same way that corner registration circles are measured. Since the interior of each target is symmetrical, we use the center-of-mass technique using a circle tool on the outer circle of the target.

The second step in measurement is to measure the location of each of the points of impact. This is done using the circle tool set to the size of the projectile used in the experiment. While the analysis has been designed for 20 shots, the tools can clearly be used regardless of the number of shots fired. While the center-of-mass technique used for the registration circles and the reference image can give sub-pixel accuracy, the circle tool will only read to the nearest pixel. Thus, when the errors are combined, the measurement accuracy for the center of the POI is about 0.07 mm. This has been found to be adequate even for air rifle work at 10 yards.

4.4. Geometrical correction. There are four types of geometrical corrections that must be made before images from a set of test shots can be compared with those from the reference image. Care should be taken in placing the paper on the scanner so that the long side of the paper is closely aligned with the scan direction.

Translation: Because there is no guarantee that the paper will be placed on the scanner in the same location each time, the images must be translated to a common reference point. We have chosen this point as the average of the measures of the centers of the four corner circles. Both the reference and the test image are translated by subtracting (\bar{x}, \bar{y}) from each measurement. A consequence of this choice is that while positive x is to the right, positive y is toward the bottom of the image.

Scaling: As with translation, there is no guarantee that the scale of each image will be exactly the same, although they should be close. For example, thermal variations among scans taken at substantially

different temperatures could lead to different scales because of thermal expansion. Thus, we scale each image based on the distances between the circles on the sides and at the top and bottom. In general, this effect is very small – a few hundredths of a percent.

Rotation: Just as the location of the paper on the scanner can not be assured, its angle relative to the scan is uncertain. We determine the angle of the paper by again using the registration circles at the corners and determining the average of the angles of the two long sides of the target. We have chosen this rather than the top and bottom because we have observed greater variation in the angles of the top and bottom, presumably because the arm of the scanner is being drawn from the top to the bottom of the scanner by a motor at one side and is rotating slightly as it is drawn down the scanner. Because these angles are small, typically a few thousandths of a radian, they are approximately the inverse of the slopes of these lines. Once the angle, θ , is known, the measured points are rotated so that the sides are approximately vertical via

$$\begin{aligned}x' &= x \cos(\theta) - y \sin(\theta) \\y' &= x \sin(\theta) + y \cos(\theta)\end{aligned}$$

where the primed coordinates are the values of x and y after the rotation. This angular correction is applied to both the reference and the test image so that they can be compared at the same angle.

Conversion to millimeters: The distance between the measured point-of-impact in the test image and the aim point from the reference image is calculated in pixels. It is then converted to millimeters via x and y scale factors. These are calculated from the known distances between the corner circles. In the case of the target shown, the centers for the horizontal circles are 160 mm apart and the vertical circles are 200 mm apart. Since the vertical and horizontal scales may not be exactly the same, we do the scaling for each axis independently.

We have checked the accuracy of this processes by making measurements of the reference target and a second target printed and scanned separately. Each of these processes used the circle tool in ImageJ to hand read the location of the central red spot in the target to the nearest pixel. In both cases, the effective CEP was ≈ 0.07 mm - much smaller than any of the CEPs we are measuring in our examples.

We have implemented these geometrical corrections in an OpenOffice [13] spreadsheet that illustrates the technique. This is available from [14]. The details of how we did the measurements are described in Appendix 2.

5. Sample results

This measurement technique, using all the data available from the shots fired, can be used to decide among options for equipment. We will illustrate the approach by examining a range of pellets for an air rifle and by comparing three different factory loads for a .308. In both of these examples it should be remembered that these results are only known to be true for the specific rifles tested. They should *not* be interpreted as applying to other rifles. Rather, they are offered as examples of the types of studies that are possible.

5.1. Air Rifle. These experiments were done using an Air Force®Talon SS with a 24 inch long, 0.22 barrel. An MTC®Viper scope set at a magnification of 13 was used for sighting. This rifle uses a spring with variable tension to throw a hammer at the valve. The spring setting used for these experiments was 5. Shots were fired from a bench using a bipod to support the front of the rifle and a Caldwell®Rock to support the rear of the rifle. Twenty shots of each pellet type were fired at 10 yards on an indoor range into the target described above and then analyzed for point-of-impact. Prior to each series of shots, the standard Air Force tank was pumped up to 200 bar as measured on an FX®airguns pump. While it is probable that the gauge on the pump is not particularly accurate; however, it should be repeatable.

Pellet	σ_x [mm]	σ_y [mm]	CEP[mm] Rayleigh	CEP [mm] RAND	t-test	$F(\sigma_x)$	$F(\sigma_y)$
JSB Exact Jumbo Express	0.87	0.95	1.07	1.07			
Crossman Premier	1.07	1.18	1.29	1.32	0.100	0.136	0.424
Beeman Kodiak	1.56	1.43	1.79	1.73	0.001	0.014	0.082
JSB Exact Jumbo	1.76	1.65	1.97	1.91	0.001	0.003	0.020
H & N Match	3.35	4.22	4.53	4.40	0.000	0.000	0.000

TABLE 4. Comparison of CEP for a range of air rifle pellets. We used a t-test to compare all the pellets to the JSB Exact Jumbo Express. F is the value from an F-test representing the probability that the two samples were drawn from populations with a common variance. In both cases, a value less than 0.1 indicates that we reject the null hypothesis at the 90% confidence level.

Cartridge	σ_x [mm]	σ_y [mm]	CEP[mm] Rayleigh	CEP [mm] RAND	t-test	$F(\sigma_x)$	$F(\sigma_y)$
Federal Sierra MatchKing	5.59	5.08	5.88	6.17			
Hornady A-MAX	6.13	5.32	7.00	6.47	0.137	0.691	0.846
Black Hills BTHP	9.07	9.36	11.03	10.84	0.000	0.041	0.011

TABLE 5. Comparison of CEP for two different types of .308 Winchester ammunition in a Savage 12 F/TR rifle. The low values returned by the F-test mean that we reject the null hypotheses that these samples have the same standard deviation at better than the 90% confidence level.

Five types of pellets were used in this comparison: JSB®Exact Jumbo, JSB®Exact Jumbo Express, Haendler & Natermann®Match-Kugeln, Crossman®Premiers and Beeman®Kodiak. They were fired dry (that is not oiled) straight from the tins without any preparation.

Table 4 shows the results of these measurements. As a simple test of whether the differences in CEP shown for these pellets are statistically significant, we have performed both a t-test on the Rayleigh CEPs and F-tests comparing the x and y variances for each pellet when compared with the variances for the JSB Exact Jumbo Express – the pellet having the smallest CEP. Using these tests, we fail to reject the null hypothesis that the Premiers are statistically indistinguishable from the JSB Exact Jumbo Express pellets. However, we reject the null hypothesis that the remaining pellets have the same mean radius or variance and hence the same CEP as the JSB Exact Jumbo Express. The small CEP for the JSB Exact Jumbo Express and the Premiers appears to be better than the others in a statistically significant way. When the CEP for the JSB Exact Jumbo Express is converted to angular measure, we see that this system has a CEP of 0.51 [moa].

5.2. .308 Winchester. Results similar to those shown above for an air rifle can also be measured for a standard firearm. In this case, our study used a Savage®12 F/TR in .308 Winchester for the experiment. Sighting was done using a Nightforce®12–42 × 56 NSX scope set at a magnification of 22. All shots were fired at 100 yards in very still air. The rifle was supported at the front by a Caldwell®Rock and in the rear by a Caldwell®sandbag.

The ammunition in this test was Federal®Premium Gold Medal Match 168 grain Sierra Matchking, in 308 Win, Hornady®A-MAX 168 grain .308 Win and Black Hills Ammunition®.308 Win Match 168 grain Boat-Tail Hollow Point. As before, 20 shots were fired for each make of cartridge. The results are shown in Table 5.

As was the case with the air rifle pellets, there is a statistically significant preference for the Federal and the Hornady ammunition over the Black Hills. The CEP for this system using the Federal ammunition, when measured in moa is approximately 0.28 [moa].

6. Conclusions

These examples illustrate the types of choices that can be made in statistically meaningful ways once quantitative measures of POI and system variability are in hand. Simple tools – a scanner and freely available software – enable a serious shooter to study different equipment configurations and make choices that improve the performance of his equipment.

References

- [1] Press, W.H., Flannery, B.P., Teukolsky, S.A., Vetterling, W.T, Numerical Recipes in C, (1988) p. 550.
- [2] RAND-234 (1952), *Offset Circle Probabilities*, Rand Corporation, 1700 Main Street, Santa Monica, CA. This document is available on the web at <http://www.rand.org/pubs/reports/2008/R234.pdf>
- [3] Williams, C.E., *A Comparison of Circular Error Probable Estimators for Small Samples*, Masters Thesis, AFIT/GOA/ENS/97M-14 (1997). This document is available on the web at <http://www.dtic.mil/cgi-bin/GetTRDoc?AD=ADA324337&Location=U2&doc=GetTRDoc.pdf>
- [4] Harter, L.H., *Circular Error Probabilities*, Journal of the American Statistical Association, Vol. 55, No. 292, (1960), pp. 723-731.
- [5] Moranda, P.B., *Comparison of Estimates of Circular Probable Error*, Journal of the American Statistical Association, Vol. 54, No. 288, (1959), pp. 794-800.
- [6] Kamat, A.R., *Some more Estimates of Circular Probable Error*, Journal of the American Statistical Association, Vol. 57, No. 297, (1962), pp. 191-195.
- [7] Elder, R.L., *An Examination of Circular Error Probable Approximation Techniques*, Masters Thesis, AFIT/GST/ENS/86-M6, (1986). This document is available on the web at <http://www.dtic.mil/cgi-bin/GetTRDoc?AD=ADA172498&Location=U2&doc=GetTRDoc.pdf>
- [8] Puhk, P., *A Sensitivity Analysis of Circular Error Probable Approximation Techniques*, Masters Thesis, AFIT/GOR/ENS/92M-23, (1992). This document is available on the web at <http://www.dtic.mil/cgi-bin/GetTRDoc?AD=ADA248105&Location=U2&doc=GetTRDoc.pdf>
- [9] <http://www.6mmbr.com/ontargetsoftware.html>
- [10] <http://www.statshooting.com/>
- [11] Rasband, W.S., ImageJ, U. S. National Institutes of Health, Bethesda, Maryland, USA, <http://rsb.info.nih.gov/ij/>, 1997-2008. Version 1.40a
- [12] As an example, Picasa from Google is freely available – <http://picasa.google.com/> – and can manage scanning the image and saving it in appropriate formats.
- [13] <http://www.openoffice.org/> – a freely available office suite including a very effective spreadsheet Version 2.4.1
- [14] <http://www.statshooting.com/tools/Metrology%20Template.ods>
- [15] http://en.wikipedia.org/wiki/Circular_error_probable

Appendix 1 – Glossary and notes

This section contains notes that may be useful to some readers in understanding a few of the more technical aspects of this paper.

- **Correlation:** A measure of how much variables vary together. If y tends to get larger when x gets larger, we say they are positively correlated. The correlation coefficient r is given by

$$r = \frac{\sum_{i=1}^N (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^N (x_i - \bar{x})^2} \sqrt{\sum_{i=1}^N (y_i - \bar{y})^2}}$$

- **F-test:** A statistical test to determine the probability that two samples of a variable are drawn from distributions having the same underlying standard deviation. In our spreadsheets, we use the FTEST as implemented in OpenOffice. The form is FTEST(data-set-1; data-set-2). The value returned is the probability that the two samples were drawn from distributions having the same standard deviation. Values less than 0.1 differ at the 90% confidence level.
- **Gaussian Distribution:** This is a statistical distribution also called a “normal distribution” or “bell curve”. In this distribution it is more likely that occurrences will be close to the mean than far from it.
- **Hypothesis testing:** The standard statistical approach to drawing inferences. In general, it involves forming a “null hypothesis”, H_0 , that is tested with a statistical test such as an F-test or a t-test. The outcomes of such a test are of the form, “fail to reject the null hypothesis at the 90% confidence level” or “reject the null hypothesis at the 90% confidence level”. While these types of statements sound like weasel words, they are the form that statisticians use. In general, our null hypothesis will be of the form that two variables from experiments are the same. We require strong evidence to reject this hypothesis. This means that if the test rejects the null-hypothesis, there is strong evidence to support that one configuration is better than the other.
- **Mean:** The average of a variable. It is defined by

$$\bar{x} = \frac{1}{N} \sum_{j=1}^N x_j$$

- **Metrology:** Measurement.
- **Monte Carlo Methods:** Monte Carlo methods use random number generators having the same properties as the system under study to generate large numbers of examples that can then be analyzed just as though they were data. While these methods are not a substitute for doing the high-powered mathematics often associated with statistics, they can often provide insight into complex systems with only modest effort.
- **Radian:** A measure of angle. π radians = 180 degrees.
- **t-test:** This is often referred to as Student’s t-test. It is a measure of whether the means of two distributions are different. We have used the implementation in OpenOffice which takes the form TTEST(data-set-1; data-set-2; 1; 3). The first two arguments are obvious. The third indicates that we are doing a 1-tailed test. The fourth indicates that the variances of the two data sets are not necessarily the same, this is referred to as being heteroscedastic. Values less than 0.1 differ at the 90% confidence level.
- **Variance:** The variance is a measure of how much a variable varies about its average value or mean – the greater the variability, the greater the variance. It is given by

$$\text{Var}(x) = \frac{1}{N} \sum_{j=1}^N (x_j - \bar{x})^2$$

The **standard deviation** is the square root of the variance.

Appendix 2 – Step-by-step measurement technique: a tutorial

There are many possible ways of implementing the technique described in this paper. For rapid prototyping we have used a spreadsheet program since none of the computations are complex. As a service to others, we have made the spreadsheet available at [14]. While this not a self-contained system, it does permit rapid analysis and easy exploration. As the technique becomes more mature, it may well be encapsulate in self-contained software.

While the approach described is in fact the way we made our measurements, it is presented in tutorial form as a simple way for others to follow our method.

Printing the target. It is probably easiest to print the target directly from the .pdf file on our web site. When you do this, be sure that scaling is turned *off* in the pdf printer. You can check this by using a pair of calipers to measure the space between the black boxes at the top and left of the target page. If these are not 100 mm, you will need to find out where the scaling is occurring and correct it. These must be the 100 mm long or all other lengths on the target will be wrong.

The source files for the target are provided as .svg files. You may choose to change the scale, number of targets etc. The system can handle this flexibility. However, if you do, remember to check the absolute lengths to be sure that they have not gotten messed up in the process.

ImageJ configuration. After installing ImageJ [11] and the Java Runtime Environment, you should be able to start ImageJ. Go to the *Analyze* menu and select *Set Measurements*. Within this tab check only the *Center of Mass* and *Centroid* boxes. Make sure no others are checked. This will ensure that when you are using the circle tool, you will get a centroid measure first and then a center of mass measure. The center of mass is calculated by weighting the location of the pixel with the intensity.

The tool that we have used for all measurements is the *Elliptical or brush selections* tool. Click on this tool. In order to force the tool to be a circle, hold down the shift key while dragging the mouse diagonally to vary the size of the circle. There will be eight “handles” on the circle that can be used to resize the circle tool. If you hold down the shift key while using the handles, the tool will remain circular when it is resized. The tool can be moved by placing the pointer inside the circle, clicking the left mouse button and dragging.

The image can be zoomed using the “+” and “-” keys on your keypad. The size of the image window can be changed by grabbing an edge or corner and dragging. Note that for the most accurate locations you may need to zoom the image to a rather large size. It is not uncommon for monitor resolution to be in the 1280×1024 domain. Thus, to be able to see each pixel in a 600 dpi scan, you will need to limit your view of the image to approximately 50 mm (2 inches) of the image.

The easiest way we have found to move around in the image is to hold down the space bar while left-clicking and dragging.

Measurements are made by hitting Ctrl-M. This opens a *Results* window. When the measurements are complete, the information in this window can be saved to a file. If the extension of this file is set to .txt, it is easily imported into the OpenOffice spreadsheet.

While ImageJ has many more tools, the ones outlined above are sufficient to make point-of-impact measurements.

Creating the reference data. To create reference data for aim points in your system use the following procedure:

- (1) Print the target page and check to be sure that the lengths are correct as above.
- (2) Scan this target page at 600 dpi in color and save the file as a jpeg file.
- (3) Start ImageJ and open the file
- (4) Convert the file to a binary file – this will change the file to strictly black or white. Do this because the printer/scanner combination will leave small flecks of white in the black regions.

You will want them to be completely black for the following operation. The following steps will give you a binary image:

- (a) Go to the *Process* menu, go down to the *Binary* entry and click on *Make Binary*. After a few seconds the image will turn to black and white. Unfortunately, this leaves the image inverted (blacks are white and vice versa) with an inverted look-up table.
 - (b) Fix the look-up table by going to the *LUT* tab and clicking on *Invert LUT*. Now the picture will look like a negative of itself.
 - (c) Now invert the image by clicking on the image window and typing Ctrl-Shift-I. Now the picture should look normal in black and white.
- (5) Measure the centers of the metrology circles by clicking on the *Elliptical or brush selection* tool. Adjust the tool as described above on one of the circles in the corners of the target pattern and adjust it so that it is a circle that fits easily within the black area of the metrology circle. Then measure each of the circles in the order shown in Figure 4. Check the results file to be sure that there are eight entries. If not, clear the entries and redo the process to this point.
 - (6) Measure the centers of each of the twenty target circles by placing the circle tool over the upper left target and expanding it until it fits easily in the black area of the large outer ring of the target. Measure the location of each target by pressing Ctrl-M. Measure left-to-right going from top to bottom. When you have completed all the targets, check the results window to be sure that there are 28 entries.
 - (7) Bring the results window to the front and click on the *Files* menu entry and save your measurements. The file should have the extension .txt so that it will open easily in the OpenOffice spreadsheet.
 - (8) If you like, you can close ImageJ at this point if you like or keep it open to measure other images.
 - (9) The metrology spreadsheet from [14] contains two tabs labeled “Reference” and “Template”. This is the spreadsheet to be used for all analysis. Open the spreadsheet with OpenOffice and go to the “Reference” tab. Use the following steps to generate the reference values:
 - (a) Go to the *Insert* menu and click on *Sheet from file*. Select the file you just created from ImageJ. You will need to answer *OK* a couple of times. Once you’ve done this, you will find a new tab in your spreadsheet that contains the data you just created.
 - (b) Click on this new spreadsheet, use the mouse to select all the data and copy it into the buffer with Ctrl-C.
 - (c) Switch back to the “Reference” spreadsheet, click on the blank square just below the box labeled “Raw Data” and paste your data into the spread sheet using Ctrl-P.
 - (d) The spreadsheet will immediately recompute the values in the table using your data. You will now find the values for your reference target in red type under the heading “Rotated”
 - (e) Use your mouse to select both columns of numbers and copy them to the buffer with Ctrl-C.
 - (f) Switch to the tab labeled “Template” and scroll right to the columns in blue headed “Reference”. Right click on the upper left number in these columns and go down to the entry marked *Paste Special*. This will give you a check box. UN-check *All* and *Formulas* and check *Numbers*. Then click *OK* to paste the numbers from the Reference table into your measurement template without putting in all the formulas.
 - (g) Delete the table in the spreadsheet that appeared when you inserted the measurement data by right clicking on the tab for that table and selecting *Delete sheet*.

You have now created a template that you can use for all measurements using this type of target. For future measurements you only need to measure the metrology circles for registration and measure points-of-impact for your projectiles.

Note that because all these measurements are made using the center-of-mass, the magnification of the image doesn’t matter. You should adjust the magnification and use the navigation approach outlined

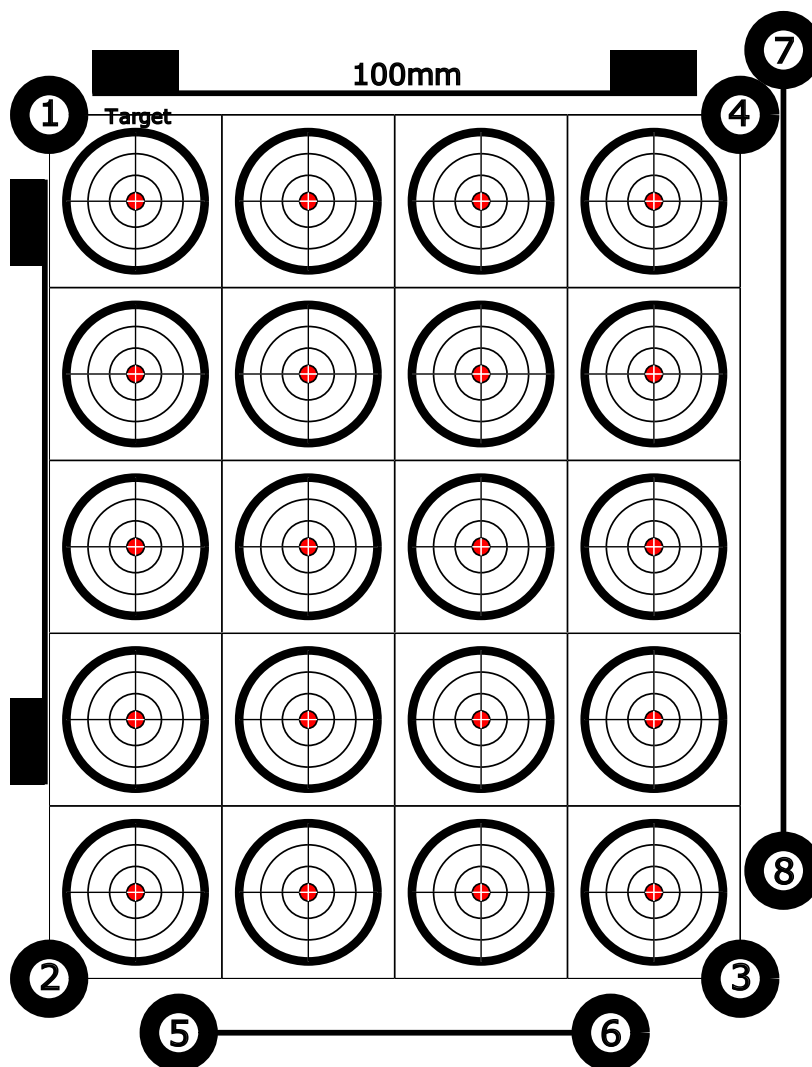


FIGURE 4. Target designed for metrology on a flatbed scanner showing the order for reading the metrology circles.

above to make it easy to position the tool.

Creating the POI data. The approach for measuring POI is similar to that outlined for the reference locations for the target. However, since there are a few differences we will detail the procedure.

This part of the process will use the “Template” sheet in the spreadsheet. The data in the upper left corner is indicative of what you may want to track in your system. You should edit this to reflect your needs. The only entries that matter for the analysis are the range and the Clicks/MOA. The range should be entered in yards (or nearly equivalently meters) and the Clicks/MOA for your scope should be entered. If your scope has 1/4 MOA clicks, enter 4, if 1/8 MOA clicks, enter 8. These are used in converting measures to minutes of arc (MOA) and scope clicks. Otherwise feel free to customize to suit your needs.

- (1) Shoot the target. Use care to parallax focus the scope and shoot as accurately as possible. While it is not necessary to shoot twenty targets, it’s a good thing to do for the statistics – you’re moving away from the perils of small number statistics. If you use fewer than 20 targets, you’ll need to modify the spreadsheet to only do the statistics on the targets you shot.
- (2) Scan the target in color at 600 dpi and save the image. Be sure to treat the target carefully until you’ve been able to scan it. Folds, damage to the metrology circles and the like could easily invalidate your measurements. We have found that it helps to see the projectile holes in the target if a piece of colored paper is put behind the target when it is scanned.
- (3) Use ImageJ as above in steps 4 and 5 to measure the eight metrology circles on a binary version of the image. Check the results file to be sure there are eight measurements.
- (4) For measuring the location of the target holes it is easiest to retrieve the color image into ImageJ. You can do this by going to the *Files* menu and choosing *Open Recent* and selecting the file you just opened. For the following measurements you may need to zoom in on the image in order to get the necessary measurement accuracy. If the holes are widely spaced, as in rifle fire at 200 yards, you probably won’t need much magnification. If on the other hand you are working with a very accurate air rifle at 10 yards, you made need substantial magnification to get the required accuracy.
- (5) Examine the holes made by the projectiles for the cleanest one that you can use to size the measurement circle. While bullets and wad-cutter pellets make reasonably clean holes in paper, many air rifle pellets leave messy holes. In any case, choose a representative hole and size the circle tool as you did in step 5 above to approximately match the size of the projectile.
- (6) Again measure the location of the POI for each target in the same sequence as you did for the target centers. It is important to keep track of the aim point for each of the holes. When a hole is a long way from the aim point, the possibility of ambiguity exists. When shooting the target, we have found it to be easy to remember which aim point goes with the hole. On retrieving the target it may make sense to annotate ambiguous holes with arrows or numbers to resolve any questions in the analysis. The holes must be scanned in the order of their aim points or the analysis will not make any sense.
 Locating the circle tool over the hole often requires a bit of judgment, particularly if the paper has torn. We have checked to see whether this causes major problems. In our analysis it has not. We have read the same target a second time to see if the results change noticeably. They change but not in a statistically significant way.
- (7) Check the “Results” window to ensure that you have 28 entries.
- (8) Save the results as before as a .txt file.
- (9) Again open the spreadsheet and create a new sheet within the spreadsheet by right clicking on one of the tabs and selecting *Insert Sheet*. Name this spreadsheet something that will help you keep track of your work, for example, date, time, type of projectile etc.
- (10) Copy the template sheet to this new sheet by going to the template sheet, clicking on the upper left box and typing Ctrl-A to select the entire sheet. Then type Ctrl-C to copy it to the buffer. Now switch back to the sheet you just created, place the cursor in the upper left box and type Ctrl-V to paste the contents of the buffer into the sheet.

- (11) Now insert the data from the measurement you just made into this spreadsheet. This is like step 9 above. Proceed as follows while in your new spreadsheet:
- (a) Go to the *Insert* menu and click on *Sheet from file*. Select the file you just created from ImageJ. You will need to answer *OK* a couple of times. Once you've done this, you will find a new tab in your spreadsheet that contains the data you just created.
 - (b) Click on this new spreadsheet, use the mouse to select all the data and copy it into the buffer with Ctrl-C.
 - (c) Switch back to the new spreadsheet, click on the blank square just below the box labeled "Raw Data" and paste your data into the spread sheet using Ctrl-P.
 - (d) The spreadsheet will immediately recompute the values in the table using your data.
 - (e) Delete the table in the spreadsheet that appeared when you inserted the measurement data by right clicking on the tab for that table and selecting *Delete sheet*.

If you have used all twenty of the targets, you will now have a summary at the top of the spreadsheet of the statistics for your shots.

Using the spreadsheet for simple analysis. In this section, we will outline the summary statistics and suggest ways to use these. To the right of the descriptive information on each spreadsheet is a box containing summary information from the analysis.

The "Mean" row is the average location of your POI. Use this to adjust your scope zero. A negative X indicates that the POI is to the left of the aim-point while positive is to the right. For the Y axis, a negative POI means that the projectile is hitting high, positive means it is low.

The "Mean [clicks]" row converts this information into scope clicks (provided your scope is properly calibrated and you've entered the correct value in cell B6). Make sure you do the corrections in the proper direction - the entries are the offset of the POI. Thus if the POI is to the left, you will need to adjust the scope to move the POI right.

The "StDev [mm]" shows the x and y standard deviations for your data in millimeters. This is a measure of the variability in your shots - smaller is better. While the mean offset can be correct by re-aiming the scope, the standard deviation is a measure of the precision of your system - it can't be reduced by adjusting the scope. It may be affected by the choice of projectile, cartridge load and the like.

The "Std. Error" row is the standard error of the means. It is given by S/\sqrt{N} where S is the sample standard deviation and N the number of data points. It represents the uncertainty in the mean measured in millimeters.

The "CEP [mm]" gives three estimates of the CEP, the two described earlier as Rayleigh and RAND, the third is the median of the data observed. These probably are the best measure to use in comparison between systems when comparing them at the same distance. The "CEP [moa]" is each of these CEPs translated into minutes-of-arc. This is useful for comparing targets shot at different distances. Further to the right are columns labeled P(90) and P(95). These represent the radii of circles in millimeters and moa respectively into which 90 and 95% of the shots are expected to fall.

There are two columns labeled C-t-c 5 and C-t-c 10. These are the set of four 5-shot c-t-c values that can be had from taking twenty shots sequentially and two 10-shot c-t-c values that can be derived from the same data. While we prefer the CEP measures for system comparisons, these c-t-c values are provided for reference because shooters are so accustomed to using them.

Care must be taken in drawing inferences. While lower CEPs are generally better, we must take care to ensure that differences we see are statistically significant. If we did the same experiment many times the CEPs we measure would change a little bit; we wouldn't expect to get exactly the same result each time. Thus, just because one result is less than another, it might not necessarily be significantly different. The t-test applied to the DR values in the spread sheet can be used to test the significance of the Rayleigh CEP while the F-test can be used on the DX and DY values in the spreadsheet to test the RAND CEP.

The subject of statistical hypothesis testing is complicated. The reader unfamiliar with these techniques is referred to a standard statistical text as a proper explanation is beyond the scope of this paper.

We have no doubt that with the POI readily available as outlined above, statistically sophisticated users will see many opportunities for hypothesis testing that go well beyond what we have presented.

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